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On the Inclinational Terms in the Moon's Coordinates.

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In the first volume of the American Journal of Mathematics, Dr. Hill published some papers in which the variation terms of the Moon's orbit were computed both literally and arithmetically with extreme accuracy. When all other terms but those depending on the ratio of the mean motions only are omitted, the remaining terms denote a periodic orbit which relatively to the Sun is a closed curve. This orbit has been made to play the part taken by the ellipse in the older lunar theories. In a subsequent paper published in the Acta Mathematica, vol. VIII, Dr. Hill obtained an expression for the motion of the Moon's perigee, so far as this depends on the ratio of the mean motions. His papers were followed by others by Prof. E. W. Brown, published in the American Journal, in which the terms depending on the solar parallax and the lunar eccentricity were computed. In all these papers Dr. Hill's orbit was treated as "the intermediary orbit," about which the actual orbit oscillates.

It is the object of this paper to take into account, according to Dr. Hill's method, the inclination of the Moon's orbit, considering it as being the manifestation of a small oscillation about Dr. Hill's distorted circular orbit, which relatively to the Sun is a closed curve.

The terms multiplied by the first power of the inclination have been calculated to the sixth order, and an expression for the part of the motion of the Moon's node, that depends upon the ratio of the mean motions only, has been found as far as the eighth order; that is to say, one term further than in Delaunay's series. The numerical values having been found to twelve places of decimals, we have the means of evaluating the residue of Delaunay's series.

The terms multiplied by the square of the inclination have been calculated to the fifth order; that is to say, to the seventh order when account is taken of the factor γ^2.

The terms multiplied by the third power of the inclination have been calculated to the fourth order in m. From these terms the part of the motion of the Moon's node that depends on the factor γ^2 has been calculated to the sixth order of m, one term further than Delaunay's series.

The principal papers dealing with Dr. Hill's Lunar Theory are tabulated in a paper by Prof. E. W. Brown, American Journal of Mathematics, vol. XVII, p. 320. The notation of the above paper will be adopted here.

Preliminary Modifications of the Problem to be Solved.

Delaunay, in his Lunar Theory, introduced two modifications. He treated the Moon's mass as negligible in comparison with the Earth's, and the Earth's mass as negligible in comparison with the Sun's. It is well known that the first modification does not affect those terms in the final expressions for the Moon's coordinates which are known as non-parallactic; that is to say, which do not vanish when the Sun's parallax is neglected. The terms containing the first power of the solar parallax as a factor can be calculated according to the first modification and then multiplied by the ratio of the difference to the sum of the masses of the Earth and Moon. The effects of the first modification in the terms containing the second and higher orders of the Sun's parallax and the effects of the second modification—treating the Earth's mass as negligible in comparison with the Sun's—cannot be allowed for so readily, but it may be inferred that, if the solar parallax be always affected with the factor denoting the ratio of the difference to the sum of the masses of the Earth and Moon, no sensible error is introduced by the first modification, while the second modification may be adopted and its results accepted as they stand without sensible error.

For these reasons the lunar theorists, with the single exception of Hansen, have modified the problem in the above two ways before entering upon the analysis.

The Differential Equations.

Taking the axis of z perpendicular to the plane of the ecliptic, and the axes x, y moving in the plane of the ecliptic, so that the axis of x is constantly directed toward the mean place of the Sun; also taking as a special unit the time in which the mean elongation of the Moon from the Sun increases by one radian; then the effective forces for the Moon are

$$\frac{d^2x}{dt^2} - 2m\frac{dy}{dt} - m^2x,$$

$$\frac{d^2y}{dt^2} + 2m\frac{dx}{dt} - m^2y,$$

$$\frac{d^2z}{dt^2},$$

where x, y, z are the coordinates of the Moon relatively to the Earth, and m is the Sun's mean motion, or, which is the same thing owing to the units of time adopted, the ratio of the Sun's mean motion to the difference of the mean motions of the Moon and Sun.

Denoting by $\varkappa$ the mass of the Earth, and in accordance with the second modification, putting $m^2a'^3$ for the mass of the Sun alone, instead of for the sum of the masses of the Sun and Earth, where a' is the Sun's mean distance, the impressed forces are derived from a potential function Ω where

$$\Omega = \frac{\varkappa}{\sqrt{(r^2+z^2)}} + \frac{m^2a'^3}{\sqrt{(r'^2 - 2Sr' + r^2 + z^2)}} - \frac{m^2a'^3}{r'^3}\cdot S,$$

where Prof. E. W. Brown's notation, reproduced below from Amer. Jour., vol. XVII, p. 321, is adopted:

$2a'$, r', e' the major axis, radius vector and eccentricity of the Sun's orbit, supposed elliptic;
x, y, z the coordinates of the Moon;
v the solar equation of the centre;
$S = x\cos v + y\sin v$;
$n'^2a'^3$ the mass of the Sun;
$\varkappa$ the sum of the masses of the Earth and Moon;
$r^2 = x^2 + y^2$.

We shall subsequently introduce

$$u = x + y\sqrt{-1}, \quad s = x - y\sqrt{-1},$$

$$\zeta = e^{(t-t_0)\sqrt{-1}}, \qquad D = \zeta\frac{d}{d\zeta} = \frac{1}{\sqrt{-1}}\frac{d}{dt}.$$

Forming the equations of motion, and taking the centrifugal terms over to the right-hand side, we have

$$\ddot{x} - 2m\dot{y} = \frac{\partial\Omega'}{\partial x},$$

$$\ddot{y} + 2m\dot{x} = \frac{\partial\Omega'}{\partial y},$$

$$\ddot{z} \qquad = \frac{\partial\Omega}{\partial z},$$

where

$$\Omega' = \Omega + \tfrac{1}{2}m^2(x^2 + y^2) = \Omega + \tfrac{1}{2}m^2r^2.$$

Transforming to the complex variables just defined, the equations become

$$D^2u + 2m\,Du = -\,2\frac{\partial\Omega'}{\partial s},$$

$$D^2s - 2m\,Ds = -\,2\frac{\partial\Omega'}{\partial u},$$

$$D^2z \qquad = -\,\frac{\partial\Omega'}{\partial z},$$

where

$$\Omega' = \frac{\kappa}{(us + z^2)^{\frac{1}{2}}} + m^2a'^3\left\{(r'^2 - 2Sr' + us + z^2)^{-\frac{1}{2}} - S.r'^{-3}\right\} + \tfrac{1}{2}m^2us$$

$$= \frac{\kappa}{(us + z^2)^{\frac{1}{2}}} + \tfrac{1}{2}m^2us + m^2\frac{a'^3}{r'^3}\;\frac{3S^2 - us - z^2}{2}$$

$$+ \frac{m^2}{a'}\;\frac{a'^4}{r'^4}\;\frac{5S^3 - 3S(us + z^2)}{2} + \ldots\ldots,$$

omitting the term $\frac{m^2a'^3}{r'}$ which disappears on differentiation.

From Ω' we select those terms that remain when the solar eccentricity (and therefore the solar equation of the centre) and also the solar parallax are put zero. In this case $S = x = (u + s)/2$, and the terms selected are

$$\frac{\kappa}{(us + z^2)^{\frac{1}{2}}} + \tfrac{3}{8}m^2(u + s)^2 - \tfrac{1}{2}m^2z^2.$$

Following Prof. E. W. Brown, we put

$$\Omega' = \frac{\kappa}{(us + z^2)^{\frac{1}{2}}} + \tfrac{3}{8}m^2(u + s)^2 - \tfrac{1}{2}m^2z^2 + \tfrac{1}{2}\Omega_1,$$

where

$$\Omega_1 = 3m^2\left[\frac{a^3}{r'^3}S^2 - \tfrac{1}{4}(u+s)^2\right] - m^2(us+z^2)\left(\frac{a'^3}{r'^3} - 1\right)$$
$$+ \frac{m^2}{a'}\cdot\frac{a'^4}{r'^4}\left[5S^3 - 3S(us+z^2)\right] + \ldots .$$

The terms which will be investigated in this paper are those that arise from treating the third coordinate of the Moon as small in comparison with the other two, instead of entirely neglecting it, as Dr. Hill did, in investigating his variation curve. The solar parallax and eccentricity will be neglected, and consequently Ω_1 is neglected also. The differential equations therefore become

$$\left(D^2 + 2mD + \tfrac{3}{2}m^2\right)u + \tfrac{3}{2}m^2 s - \frac{\varkappa u}{(us+z^2)^{\frac{3}{2}}} = 0,$$
$$\left(D^2 - 2mD + \tfrac{3}{2}m^2\right)s + \tfrac{3}{2}m^2 u - \frac{\varkappa s}{(us+z^2)^{\frac{3}{2}}} = 0,$$
$$\left(D^2 - m^2 \qquad\qquad\right)z \qquad\qquad - \frac{\varkappa z}{(us+z^2)^{\frac{3}{2}}} = 0.$$

When z is neglected, the solution involves four arbitrary constants: of these, two, viz. the linear dimensions of the orbit* and the time of conjunction, occur in Dr. Hill's investigations of the variation curve. This curve represents a periodic motion, but inasmuch as two arbitrary constants are still wanting, it is not the most general solution of the equations in their modified form. The general solution should be regarded as denoting an oscillation superimposed upon Dr. Hill's periodic motion. The constants introduced define the amplitude of this oscillation and its phase. They replace the lunar eccentricity and the time of passage through perigee of the theories that take as the intermediary the ellipse and not the variation curve. As however in this paper we do not propose to retain the lunar eccentricity, we quote from Dr. Hill's papers (Amer. Jour., vol. 1) the particular solution of the differential equations

$$z = 0,$$
$$u = u_0 = a_0\Sigma a_i\zeta^{2i+1},$$
$$s = s_0 = a_0\Sigma a_{-i-1}\zeta^{2i+1},$$

---

* m may have any arithmetic value consistent with convergence. Hence it appears that the mean motion and the mean distance are both arbitrary. This is because κ, the mass of the Earth and Moon, will be eliminated before integration. It follows that κ is a definite function of m and a_0 (the linear dimension), and possibly of other constants as well.

the algebraical and numerical values of the quantities a_i being given in the papers referred to.

Let γ denote a constant of the same order as z. Then we are seeking the most general solution of the form

$$z = z_{\gamma} + z_{\gamma^3} + z_{\gamma^5} + \dots,$$
$$u = u_0 + u_{\gamma^2} + u_{\gamma^4} + \dots,$$
$$s = s_0 + s_{\gamma^2} + s_{\gamma^4} + \dots,$$

where the suffixes denote the order of the terms in γ.

The most general solution should contain six arbitrary constants. Two, however, disappear, as we are not considering the lunar eccentricity. We should therefore still have four. Besides the two introduced by Dr. Hill, we shall have one, K, replacing Delaunay's γ, defining the amplitude of the oscillation about the variation curve and another defining the phase of that oscillation at a given epoch.

It is useful to regard the terms involving K as defining an oscillation about a periodic motion, for we are thereby enabled to predict the character of the solution. So long as the oscillation is small, so that the square of its amplitude may be neglected, we know that its period is independent of its amplitude. The principal part of the motion of the Moon's node is therefore a function of m only. Considerations of symmetry show that u, s will involve only even powers and z only odd powers of K in accordance with the type of solution assumed above. The period of the oscillation, more commonly called the period of revolution with respect to the node, will clearly only involve even powers of K.

The First Power of the Inclination.

We must try and satisfy the differential equations accurately to order K with the assumed solutions

$$u = u_0, \quad s = s_0, \quad z = z_{\gamma}.$$

The first two equations are obviously satisfied. The third equation, on neglecting K^3, becomes

$$D^2 z_{\gamma} = \left(m^2 + \frac{\varkappa}{r^3}\right) z_{\gamma}.$$

The solution is of the form

$$z_\gamma\sqrt{-1} = Ka_0\sum_{-\infty}^{\infty}K_j\left(\zeta^{2j+g} - \zeta^{-2j-g}\right),$$

where $K_0 = 1$.

The quantities K_j are all determinate: K is one of the two arbitrary constants required: the other has not been put in evidence, but corresponding to the definition $\zeta = e^{\sqrt{-1}(t-t_0)}$, we may suppose $\zeta^g = e^{\sqrt{-1}g(t-t_1)}$, in which case t_1 is the second arbitrary constant required, defining the time of passage through the node. No confusion will be introduced by these definitions, since g is incommensurable, and integral powers only of ζ and ζ^g are required.

The expression $m^2 + \frac{\varkappa}{(u_0s_0)^{\frac{3}{2}}}$ involves only quantities calculated by Dr. Hill, and which we therefore suppose known.

Following Prof. E. W. Brown's notation, we write

$$\tfrac{1}{2}m^2 + \tfrac{1}{2}\frac{\varkappa}{(u_0s_0)^{\frac{3}{2}}} = \sum_{-\infty}^{\infty}M_i\zeta^{2i}.$$

The quantities M_i are calculated arithmetically by the method of special values, and algebraically by a process explained by Dr. Hill (Acta Math., vol. VIII, p. 12). Dr. Hill has not however obtained the algebraical values to the order required for this paper. By his methods the following results have been obtained:

$$2M_0 = 1 + 2m + \frac{5}{2}m^2 + 0.m^3 - \frac{9}{2^5}m^4 + 4m^5 + \frac{34}{3}m^6 + 15m^7 + \frac{2704801}{2^{13}.3^3}m^8,$$

$$2M_1 = \frac{3}{2}m^2 + \frac{19}{2^2}m^3 + \frac{20}{3}m^4 + \frac{43}{3^2}m^5 + \frac{18709}{2^9.3^3}m^6 + \frac{759413}{2^{10}.3^4.5}m^7,$$

$$2M_2 = \frac{33}{2^4}m^4 + \frac{2937}{2^6.5}m^5 + \frac{23051}{2^4.3.5^2}m^6,$$

$$2M_3 = \frac{1393}{2^9}m^6.$$

Generally M_i is of order m^{2i}.

Substituting the assumed solution in the differential equation, we obtain

$$(2j+g)^2K_j - 2\sum_{i=-\infty}^{\infty}K_iM_{j-i} = 0.$$

By giving j the values $0, \pm 1, \pm 2, \ldots . \pm k$, we obtain $2k+1$ equations which will determine the ratios of K_j to K_0 $(j = \pm 1, \ldots . \pm k)$ and also the value of g. In the equation above written the coefficient of K_j is $(2_j+g)^2-2M_0$ in general, a large quantity, whereas every other coefficient is of the second order at least. The equation therefore serves primarily to determine K_j. If we put $j=0$ we get

$$g^2 = 2\sum_{i=-\infty}^{\infty} M_{-i} K_i / K_0,$$

an equation for determining g.

From the last equation we obtain

$$g^2 = 2M_0 \text{ to order } m^2,$$

whence

$$g = 1 + m + \frac{3}{2^2} m^2.$$

Hence, whereas $(2j+g)^2 - 2M_0$ is generally of order m^0, when $j=-1$ it is of order m. The equation for K_{-1},

$$[(-2+g)^2 - 2M_0]\, K_{-1} = 2\sum_{\substack{-\infty \\ i \neq -1}}^{\infty} K_i M_{-1-i},$$

shows that $K_{-1} : K_0$ is of order m. The general law that $K_{-k} : K_0$ is of order m^{2k-1}, and that $K_k : K_0$ is of order m^{2k} now follows easily.

Proceeding by continued approximation, we obtain

$$g = 1 + m + \frac{3}{2^2} m^2 - \frac{33}{2^5} m^3 - \frac{105}{2^7} m^4 + \frac{43}{2^{11}} m^5 + \frac{2567}{2^{13}.3} m^6 + \frac{347699}{2^{16}.3^2} m^7 + \frac{6442309}{2^{18}.3^3} m^8,$$

and the subsidiary quantities

$$g^2 = 1 + 2m + \frac{5}{2} m^2 - \frac{9}{2^4} m^3 - \frac{201}{2^6} m^4 - \frac{3221}{2^{10}} m^5 + \frac{1031}{2^{12}.3} m^6 + \frac{917555}{2^{15}.3^2} m^7 + \frac{13397317}{2^{17}.3^3} m^8,$$

$$g^2 - 2M_0 = -\frac{9}{2^4} m^3 - \frac{183}{2^6} m^4 - \frac{7317}{2^{10}} m^5 - \frac{138233}{2^{12}.3} m^6 - \frac{3506125}{2^{15}.3^2} m^7 - \frac{9959833}{2^{17}.3^2},$$

$$\begin{aligned}
(-2+g)^2-2M_0 &= -4m-3m^2+\frac{57}{2^4}m^3+\frac{27}{2^6}m^4-\frac{7403}{2^{10}}m^5-\frac{47789}{2^{12}}m^6,\\
(2+g)^2-2M_0 &= 8+4m+3m^2-\frac{75}{2^4}m^3-\frac{393}{2^6}m^4,\\
(-4+g)^2-2M_0 &= 8-8m-6m^2+\frac{123}{2^4}m^3,\\
(4+g)^2-2M_0 &= 24+8m+6m^2,\\
(-6+g)^2-2M_0 &= 24-12m,\\
(6+g)^2-2M_0 &= 48,
\end{aligned}$$

and also the coefficients

$$\begin{aligned}
K_{-1}:K_0 &= -\frac{3}{2^3}m-\frac{29}{2^5}m^2-\frac{2029}{2^9.3}m^3-\frac{18875}{2^{11}.3^2}m^4-\frac{44633}{2^{14}.3^3}m^5+\frac{5569883}{2^{16}.3^4}m^6,\\
K_1:K_0 &= \frac{3}{2^4}m^2+\frac{1}{2}m^3+\frac{197}{2^7.3}m^4+\frac{3067}{2^{11}.3^2}m^5-\frac{348449}{2^{13}.3^3.5}m^6,\\
K_{-2}:K_0 &= -\frac{9}{2^7}m^3-\frac{105}{2^9}m^4-\frac{8553}{2^{13}.5}m^5+\frac{123367}{2^{15}.5^2}m^6,\\
K_2:K_0 &= \frac{25}{2^8}m^4+\frac{803}{2^7.3.5}m^5+\frac{377701}{2^{11}.3^2.5^2}m^6,\\
K_{-3}:K_0 &= -\frac{75}{2^{11}}m^5-\frac{18797}{2^{13}.3.5}m^6,\\
K_3:K_0 &= \frac{833}{2^{12}.3}m^6.
\end{aligned}$$

Comparison with Delaunay's Expressions.

From Dr. Hill's expressions for x and y we form $(x^2+y^2)/a_0^2$, and hence $a_0(x^2+y^2)^{-\frac{1}{2}}$. Multiplying by z/a_0, we get an expression for the tangent of the latitude, or correctly to the first power of the inclination the latitude itself. This expression for the latitude is

$$\begin{aligned}
2K\Big[&\left(1-\frac{3}{2^4}m^3-\frac{141}{2^8}m^4-\frac{3893}{2^{10}.3}m^5-\frac{77345}{2^{12}.3^2}m^6\right)\sin F\\
&+\left(\frac{11}{2^4}m^2+\frac{13}{2^2.3}m^3+\frac{943}{2^7.3^2}m^4+\frac{6017}{2^{11}.3^3}m^5-\frac{3148129}{2^{13}.3^4.5}m^6\right)\sin(2D+F)\\
&+\left(\frac{161}{2^8}m^4+\frac{3607}{2^7.3.5}m^5+\frac{1282109}{2^{11}.3^2.5^2}m^6\right)\sin(4D+F)\\
&+\frac{7697}{2^{12}.3}m^6\sin(6D+F)
\end{aligned}$$

$$+\left(-\frac{3}{2^3}m-\frac{13}{2^5}m^2-\frac{1133}{2^9.3}m^3-\frac{13243}{2^{11}.3^2}m^4\right.$$
$$\left.-\frac{17617}{2^{14}.3^3}m^5+\frac{6400195}{2^{16}.3^4}m^6\right)\sin(-2D+F)$$
$$+\left(-\frac{33}{2^7}m^3-\frac{225}{2^9}m^4-\frac{50563}{2^{13}.3.5}m^5-\frac{75041}{2^{15}.5^2}m^6\right)\sin(-4D+F)$$
$$\left.+\left(-\frac{483}{2^{11}}m^5-\frac{83333}{2^{13}.3.5}m^6\right)\sin(-6D+F)\right].$$

This is easily seen to be identical with Delaunay's expression (Delaunay's m denotes the ratio of the mean motions), provided that we equate Delaunay's γ to $K\left[1-\frac{3}{2^4}m^3-\frac{141}{2^8}m^4-\frac{3893}{2^{10}.3}m^5-\frac{77345}{2^{12}.3^2}m^6\right]$. This result is necessary to enable us to compare the two theories.

Alternative Calculation of g.

The quantity g may be determined alone without simultaneously calculating the quantities K. The method pursued is analogous to that in Dr. Hill's paper on the motion of the perigee (Acta Math., vol. VIII).

If we eliminate the quantities K from the equations

$$[(2j+g)^2-2M_0]K_j-2\sum_{\substack{i=-\infty\\ i\neq j}}^{\infty}K_iM_{j-i}=0$$

we obtain a determinant of an infinite number of rows and columns.

$$0=\begin{vmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & 1, & \frac{-2M_1}{(4+g)^2-2M_0}, & -\frac{2M_2}{(4+g)^2-2M_0}, & -\frac{2M_3}{(4+g)^2-2M_0}, & -\frac{2M_4}{(4+g)^2-2M_0} & \cdots\\
\cdots & -\frac{2M_1}{(2+g)^2-2M_0}, & 1, & -\frac{2M_1}{(2+g)^2-2M_0}, & -\frac{2M_2}{(2+g)^2-2M_0}, & -\frac{2M_3}{(2+g)^2-2M_0} & \cdots\\
\cdots & -\frac{2M_2}{g^2-2M_0}, & -\frac{2M_1}{g^2-2M_0}, & 1, & -\frac{2M_1}{g^2-2M_0}, & -\frac{2M_2}{g^2-2M_0} & \cdots\\
\cdots & -\frac{2M_3}{(-2+g)^2-2M_0}, & -\frac{2M_2}{(-2+g)^2-2M_0}, & -\frac{2M_1}{(-2+g)^2-2M_0}, & 1, & -\frac{2M_1}{(-2+g)^2-2M_0} & \cdots\\
\cdots & -\frac{2M_4}{(-4+g)^2-2M_0}, & -\frac{2M_3}{(-4+g)^2-2M_0}, & -\frac{2M_2}{(-4+g)^2-2M_0}, & -\frac{2M_1}{(-4+g)^2-2M_0}, & 1 & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix}$$

This equation determines g. Its use, according to M. Poincaré, is legitimate, since the sum of the elements not situated on the leading diagonal is finite, and the determinant therefore converges absolutely and uniformly.

One solution of the equation

$$D^2 z = \left(\frac{\varkappa}{r^3} + m^2\right) z$$

is

$$z = \Sigma K_j \zeta^{2j+g}.$$

This solution may be written

$$z = \sum_j K'_j \zeta^{2j+(g+2i)}$$

when

$$K'_j = K_{i+j}.$$

In this second form of the solution, $g + 2i$ replaces g, and consequently we conclude that if g is a root of the determinantal equation, $g + 2i$ is also a root when i is any integer, positive or negative. This result also follows from an inspection of the determinant. Since also the determinantal equation is unaltered on changing the sign of g, we conclude that if g_0 be one root, $\pm 2i \pm g_0$ (i, integral) is also a root. There can be no root not included under this expression, for else there would be a third solution of the third differential equation of motion linearly independent of the two known solutions

$$z = \Sigma K_j \zeta^{2j+g}$$

and

$$z = \Sigma K_j \zeta^{-2j-g},$$

which is impossible for a linear differential equation of the second order.

Since the roots are not repeated when we limit ourselves to a finite number of rows and columns, we infer that the roots are not repeated in the case considered. The infinite determinant must therefore be identically equal to

$$A\,(\cos \pi g - \cos \pi g_0)/(\cos \pi g_0 - \cos \pi \sqrt{2M_0}),$$

the denominator representing to a numerical factor the product of the denominators in the rows of the infinite determinant. When a finite number of rows are considered, the coefficient of the highest power of g is independent of the quantities M_i, and we infer therefore that so is A. On putting $M_i = 0$ $(i \neq 0)$, we then obtain $A = 1$. Finally we put $g = 0$ in the identity, and we obtain

$$\frac{\sin^2 \frac{\pi}{2} g_0}{\sin^2 \frac{\pi}{2} \surd\, 2M_0} = \Box(0) =$$

$$\left|\begin{array}{ccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 1\;, & -\frac{2M_1}{4^2-2M_0}, & -\frac{2M_2}{4^2-2M_0}, & -\frac{2M_3}{4^2-2M_0}, & -\frac{2M_4}{4^2-2M_0}, & \cdots \\
\cdots & -\frac{2M_1}{2^2-2M_0}, & 1\;, & -\frac{2M_1}{2^2-2M_0}, & -\frac{2M_2}{2^2-2M_0}, & -\frac{2M_3}{2^2-2M_0}, & \cdots \\
\cdots & -\frac{2M_2}{0^2-2M_0}, & -\frac{2M_1}{0^2-2M_0}, & 1\;, & -\frac{2M_1}{0^2-2M_0}, & -\frac{2M_2}{0^2-2M_0}, & \cdots \\
\cdots & -\frac{2M_3}{2^2-2M_0}, & -\frac{2M_2}{2^2-2M_0}, & -\frac{2M_1}{2^2-2M_0}, & 1\;, & -\frac{2M_1}{2^2-2M_0}, & \cdots \\
\cdots & -\frac{2M_2}{4^2-2M_0}, & -\frac{2M_3}{4^2-2M_0}, & -\frac{2M_2}{2^2-2M_0}, & -\frac{2M_1}{2^2-2M_0}, & 1\;, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right|$$

In this equation we may drop the suffix after g_0.

Dr. Hill has developed a determinant to $\Box(0)$ (Acta Math., vol. VIII, p. 29–31). Using his results, but suppressing terms of order m^{10}, we obtain

$$\Box(0) = 1 + \frac{\pi \cot\left(\frac{\pi}{2}\surd(2M_0)\right)}{4\surd(2M_0)}\left\{\frac{4M_1^2}{1-2M_0} + \frac{4M_2^2}{4-2M_0}\right\}$$

$$+ \frac{\pi \cot\left(\frac{\pi}{2}\surd(2M_0)\right)}{32\surd(2M_0)(1-2M_0)^2}\left\{\frac{\pi\cot(\pi\surd(2M_0))}{\surd(2M_0)} - \frac{1}{2M_0} + \frac{2}{1-2M_0} + \frac{9}{2.(4-2M_0)}\right\}(2M_1)^4$$

$$+ \frac{3\pi \cot\left(\frac{\pi}{2}\surd(2M_0)\right)}{8\surd(2M_0)(1-2M_0)(4-2M_0)}(2M_1)^2 .\, 2M_2$$

$$+ \frac{\pi \cot\left(\frac{\pi}{2}\surd(2M_0)\right)}{128\surd(2M_0)(1-2M_0)^3}\left\{\left(-\frac{1}{2M_0} + \frac{2}{1-2M_0} + \frac{9}{2.(4-2M_0)}\right)\frac{\pi\cot\pi\surd(2M_0)}{\surd(2M_0)}\right.$$

$$- \frac{25}{16M_0} + \frac{1}{4M_0^2} + \frac{2}{1-2M_0} + \frac{4}{(1-2M_0)^2} - \frac{9}{8.(4-2M_0)}$$

$$\left. + \frac{9}{(4-2M_0)^2} - \frac{4}{9-2M_0} - \frac{\pi^2}{6M_0}\right\}(2M_1)^6.$$

The last term has been written down because some parts of it are apparently of the eighth order. These parts, however, cancel out, and the term is zero to the ninth order inclusive.

Now

$$\frac{\sin^2 \frac{\pi}{2} g}{\sin^2 \frac{\pi}{2} \sqrt{(2M_0)}} = \frac{1 - \sin^2 \frac{\pi}{2} (g-1)}{1 - \sin^2 \frac{\pi}{2} (\sqrt{(2M_0)} - 1)}$$

$$= 1 - \frac{\pi^2}{4} \left\{ (g-1)^2 - (\sqrt{(2M_0)} - 1)^2 \right\} + \ldots$$

Hence $(\sqrt{(2M_0)} - 1)^2 - (g-1)^2 =$ coefficient of $\frac{\pi^2}{4}$ in □ (0) developed in powers of π.

To obtain the coefficient of π^2 in □ (0),

$$\cot \frac{\pi}{2} \sqrt{(2M_0)} \text{ may be replaced by } -\frac{\pi}{2} (\sqrt{(2M_0)} - 1),$$

$$\pi \cot (\pi \sqrt{(2M_0)}) \quad \text{``} \qquad \text{``} \qquad \frac{1}{\sqrt{(2M_0)} - 1},$$

and

$$\sqrt{(2M_0)} = 1 + m + \frac{3}{2^2} m^2 - \frac{3}{2^4} m^3 + \frac{21}{2^6} m^4 + \frac{143}{2^6} m^5$$

$$+ \frac{2231}{2^8 . 3} m^6 + \frac{2431}{2^8 . 3} m^7 + \frac{1058251}{2^{14} . 3^3} m^8.$$

Hence to the coefficient of $\frac{\pi^2}{4}$ in □ (0),

$$\frac{3\pi \cot \left(\frac{\pi}{2} \sqrt{(2M_0)} \right)}{8 \sqrt{(2M_0)} (1 - 2M_0)(4 - 2M_0)} (2M_1)^2 2M_2 \text{ contributes } \frac{297}{2^9} m^8 + \frac{59103}{2^{11} . 5} m^9,$$

$$\frac{\pi \cot \left(\frac{\pi}{2} \sqrt{(2M_0)} \right)}{32 \sqrt{(2M_0)} (1 - 2M_0)^2} \left[\frac{\pi \cot \pi \sqrt{(2M_0)}}{\sqrt{(2M_0)}} - \frac{1}{2M_0} + \frac{2}{1 - 2M_0} + \frac{9}{2(4 - 2M_0)} \right]$$

$$\text{contributes } -\frac{1215}{2^{12}} m^8 - \frac{24165}{2^{13}} m^9,$$

$$\frac{\pi}{4} \frac{\cot \left(\frac{\pi}{2} \sqrt{(2M_0)} \right)}{\sqrt{(2M_0)}} \frac{4M_2^2}{4 - 2M_0} \text{ contributes } -\frac{363}{2^9} m^9,$$

$$\frac{\pi \cot \frac{\pi}{2} \sqrt{(2M_0)}}{4 \sqrt{(2M_0)}} \frac{4M_1^2}{1 - 2M_0} \text{ contributes } \frac{4M_1^2}{2 \sqrt{(2M_0)} (\sqrt{(2M_0)} + 1)}.$$

Now

$$4M_1^2 = \frac{9}{2^2}m^4 + \frac{57}{2^2}m^5 + \frac{681}{2^4}m^6 + \frac{233}{3}m^7 + \frac{432661}{2^9.3^2}m^8 + \frac{19693}{2^4.3.5}m^9,$$

$$2\surd(2M_0)(\surd(2\mathrm{M}_0)+1) = 4+6m+\frac{13}{2}m^2 - \frac{3}{2}m^3 + \frac{3}{2^5}m^4 + \frac{399}{2^5}m^5,$$

whence the term

$$= \frac{9}{2^4}m^4 + \frac{87}{2^5}m^5 + \frac{723}{2^7}m^6 + \frac{2587}{2^7.3}m^7 + \frac{23941}{2^9.3^2}m^8 + \frac{42469}{2^{12}.5}m^9.$$

Also

$$(\surd(2M_0)-1)^2 = m^2 + \frac{3}{2}m^3 - \frac{15}{2^4}m^4 - \frac{15}{2^5}m^5 + \frac{707}{2^7}m^6$$
$$+ \frac{3329}{2^7.3}m^7 + \frac{91475}{2^{12}.3}m^8 + \frac{1468975}{2^{13}.3^3}m^9.$$

Hence

$$(g-1)^2 = m^2 + \frac{3}{2}m^3 - \frac{3}{2}m^4 - \frac{51}{2^4}m^5 - \frac{1}{2^3}m^6$$
$$+ \frac{371}{2^6.3}m^7 + \frac{283}{2^4.3^2}m^8 + \frac{135739}{2^{11}.3^3}m^9,$$

and hence

$$g = 1 + m + \frac{3}{2^2}m^2 - \frac{33}{2^5}m^3 - \frac{105}{2^7}m^4 + \frac{43}{2^{11}}m^5$$
$$+ \frac{2567}{2^{13}.3}m^6 + \frac{347699}{2^{16}.3^2}m^7 + \frac{6442309}{2^{18}.3^3}m^8,$$

the same result as before.

This agrees with, and is carried one term beyond, Delaunay's result.

The coefficients in the series in powers of Dr. Hill's m are considerably smaller than the coefficients of Delaunay's series. The increase in convergency is even more striking in the expression for the latitude: for instance, the term $\gamma . m^6 \sin(-2D+F)$, which here has a coefficient $\frac{1955563}{2^{16}.3^4}$ in Delaunay's series has a coefficient $-\frac{100407473}{2^{16}.3^4}$.

Arithmetical Results.

Dr. Hill has calculated by the method of special values an expression for x/r^3 with numerical coefficients (Amer. Journ., vol. I, p. 249). We hence obtain

$$\begin{array}{llll}
2M_0 = 1. & 17804 & 45712 & 77166 \\
2M_1 = 0. & 01261 & 68462 & 48930 \\
2M_2 = 0. & 00012 & 57766 & 75006 \\
2M_3 = 0. & 00000 & 12059 & 39899 \\
2M_4 = 0. & 00000 & 00113 & 02925 \\
2M_5 = 0. & 00000 & 00001 & 04375 \\
2M_6 = 0. & 00000 & 00000 & 00954 \\
2M_7 = 0. & 00000 & 00000 & 00008
\end{array}$$

The following arithmetical values have been obtained:

$$g = 1.\ \ 08517\ \ \ 14265\ \ \ 58$$

$$\begin{array}{llllllll}
K_{-1} = -.03698 & 39313 & 94 & \qquad & K_1 = +.00151 & 22192 & 28 \\
K_{-2} = -.00004 & 65750 & 01 & & K_2 = +.00000 & 58673 & 61 \\
K_{-3} = -.00000 & 01755 & 37 & & K_3 = +.00000 & 00299 & 82 \\
K_{-4} = -.00000 & 00008 & 87 & & K_4 = +.00000 & 00001 & 75 \\
K_{-5} = -.00000 & 00000 & 05 & & K_5 = +.00000 & 00000 & 01
\end{array}$$

In performing the arithmetic, the coefficients of the quantities K were replaced by their arithmetical values. Since the arithmetical values of the quantities K were already approximately known from the series representing them, it was obvious to how many decimal places each coefficient had to be calculated.

To this value of g corresponds a motion of the node given by

$$-\frac{1}{n}\frac{d\Omega}{dt} = \frac{g-1-m}{1+m} = .00399\ \ 91645.$$

Delaunay's series gives $.00399\ \ 91772,$

a difference equal to about .00000 3 of the whole.

As the error between theory and observation amounts to .0006 of the whole, it appears that it cannot be accounted for, as Delaunay supposed, by the incompleteness of that part of his expression which depends on the ratio of mean motions only.

Adams (M. N. R. A. S., Nov. 1877) has obtained the result:

$$g = 1\ \ 08517\ \ 13927\ \ 46869.$$

The discrepancy between the value obtained in this paper and that obtained by

Adams is due to the different assumed values of the mean motions. Adams uses

$$m = 0.\ 08084\ 89030\ 52,$$

whereas in this paper Dr. Hill's value

$$m = 0.\ 08084\ 89338\ 08312$$

is employed.

It is easy to verify that the difference in the two values of g divided by the difference in the two values of m is the arithmetical value of dg/dm as calculated from the algebraic series for g.

Second Powers of the Inclination.

We assume as the solution of the differential equations

$$u = u_0 + u_{\gamma^2}, \qquad s = s_0 + s_{\gamma^2}$$

correctly to order γ^2 where

$$\zeta^{-1}.\, u_{\gamma^2} = a_0 K^2 \sum_j \left[(kk)_j\, \zeta^{2j+2g} + (kk')_j\, \zeta^{2j} + (k'k')_j\, \zeta^{2j-2g}\right],$$

and

$$\zeta\,.\, s_{\gamma^2} = a_0 K^2 \sum_j \left[(kk)_j\, \zeta^{-2j-2g} + (kk')_j\, \zeta^{-2j} + (k'k')_j\, \zeta^{-2j+2g}\right].$$

The differential equations to be satisfied are

$$(D^2 + 2m\,D + \tfrac{3}{2}\,m^2)\,u + \tfrac{3}{2}\,m^2 s - \frac{\varkappa u}{(us + z^2)^{\frac{3}{2}}} = 0,$$

$$(D^2 - 2m\,D + \tfrac{3}{2}\,m^2)\,s + \tfrac{3}{2}\,m^2 u - \frac{\varkappa s}{(us + z^2)^{\frac{3}{2}}} = 0,$$

where

$$z\sqrt{-1} = K a_0 \Sigma K_j\,(\zeta^{2j+g} - \zeta^{-2j-g})$$

and satisfies the equation

$$(D^2 - m^2)\,z - \frac{\varkappa z}{(us + z^2)^{\frac{3}{2}}} = 0.$$

The quantity $(us + z^2)^{-\frac{3}{2}}$ is inconvenient to deal with, and must be eliminated. First, multiplying the equations by s and u and subtracting, we get

$$D(-uDs + sDu + 2mus) - \tfrac{3}{2}\,m^2\,(u^2 - s^2) = 0. \qquad \text{(i)}$$

Again multiplying the equations by s, u, $2z$ and adding,

$$D^2(us) - 2m(uDs - sDu) + \tfrac{3}{2}m^2(u+s)^2 - 2Du\,Ds$$
$$+ 2zD^2z - 2m^2z^2 - \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}} = 0.$$

The last term must be eliminated by the help of Jacobian integral of relative motion, obtained by multiplying the equations by Ds, Du, $2Dz$, adding and integrating.

$$Du\,Ds + Dz^2 + \tfrac{3}{4}m^2(u+s)^2 - m^2z^2 + \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}} = C, \text{ a const.}$$

Adding this equation to the last,

$$D^2(us) - 2m(uDs - sDu) + \tfrac{9}{4}m^2(u+s)^2 - Du\,Ds = C + 3m^2z^2 - 2zD^2z - Dz^2. \quad \text{(ii)}$$

Calculation of the Quantities (kk').

If we substitute the above assumed values in the equations (i) and (ii) and equate the coefficients of ζ^{2j} to zero, we obtain the two equations

$$\sum_i (kk')_i \Big[\{4j^2 + (2i+1-2j)(2i+1) + 4m(2i+1-j) + \tfrac{9}{2}m^2\} a_{i-j}$$
$$+ \{4j^2 + (2i+1+2j)(2i+1) + 4m(2i+1+j) + \tfrac{9}{2}m^2\} a_{i+j}$$
$$+ \tfrac{9}{2}m^2 a_{j-i-1} + \tfrac{9}{2}m^2 a_{-j-i-1}\Big]$$
$$+ 2\sum_i K_{i+j}K_i\{4j^2 + 2j(2i+g) + (2i+g)^2 - 3m^2\} = 0,$$

and

$$4j\sum_i(2i+1-j+m)a_{i-j}(kk')_i + 4j\sum_i(2i+1+j+m)a_{i+j}(kk')_i$$
$$- 3m^2\sum_i a_{j-i-1}(kk')_i + 3m^2\sum_i a_{-j-i-1}(kk')_i = 0.$$

In these equations $(kk')_j$ and $(kk')_{-j}$ occur with coefficients that are large in comparison with the coefficients of the other unknown quantities. Hence the equations may be considered as determining $(kk')_j$ and $(kk')_{-j}$, and we may replace them by two others in which $(kk')_{-j}$ and $(kk')_j$ have been respectively eliminated. The equations thus obtained will be found to differ in the sign of j only, and one equation may therefore be suppressed, on the understanding that the other holds for negative as well as positive values of j. We are thus left with the following equation:

$$\begin{aligned}
&-4j\Sigma_i(j-i)[[j]-4i(-1+j-m)]\,a_{-i}(kk')_{j-i}\\
&-4j\Sigma_i i\,[[j]-4(j-i)(-1+j-m)]\,a_i(kk')_{j+i}\\
&+\frac{3m^2}{2}[[j]-4(-1+j-m)(i+3j+2m)]\,\Sigma_i a_{-i+j-1}(kk')_i\\
&+\frac{3m^2}{2}[-[j]-4(-1+j-m)(-1+3j-2m)]\,\Sigma_i a_{-i-j-1}(kk')_i\\
&-8j(-1+j-m)\Sigma\{4j^2-2j(2i+g)+(2i+g)^2-3m^2\}\,K_iK_{i-j}\\
&=0,
\end{aligned}$$

where

$$[j]=8j^2-2-4m+m^2.$$

This equation is an identity when $j=0$. Hence the number of the quantities (kk') exceeds by unity the number of equations to determine them. We may therefore assign any arbitrary value to one of them $(kk')_0$. Now

$$a_0(1+K^2(kk')_0)$$

is the mean value of the projection of the radius vector upon a line drawn towards the mean position of the Moon. a_0 has already been supposed to be equivalent to this quantity, when powers of K are neglected. Any assumption as to $(kk')_0$ is in reality equivalent to a more accurate definition of a_0.

Let us therefore assume $(kk')_0=0$, an assumption that simplifies the equations considerably, and which is equivalent to assuming that a_0 is equal to the mean value of the projection of the radius vector upon a line drawn to the Moon's mean place, accurately, even when quantities of the order of the square of the inclination are taken into account.

Note.—We might use the arbitrary condition at our disposal so that the relation between k, the mass of the Earth and Moon, and a_0 and m remain the same as that found by Dr. Hill (Acta Math., vol. I, p. 145) even in the higher order of approximation which we have now reached. Otherwise the relation referred to must now be modified by the addition of terms proportional to the square of the inclination.

Solving the equations, we obtain

$$\begin{aligned}
(kk')_{-1} &= \frac{3}{2^2}\,m + \frac{29}{2^4}\,m^2 + \frac{661}{2^8.3}\,m^3 - \frac{26125}{2^{10}.3^2}\,m^4 - \frac{1385395}{2^{13}.3^3}\,m^5,\\
(kk')_{1} &= \qquad -\frac{3}{2^3}\,m^2 - \frac{23}{2^5}\,m^3 - \frac{109}{2^9.3}\,m^4 + \frac{29857}{2^{11}.3^2}\,m^5,\\
(kk')_{-2} &= \qquad\qquad \frac{9}{2^6}\,m^3 + \frac{105}{2^8}\,m^4 + \frac{6001}{2^{12}.5}\,m^5\\
(kk')_{2} &= \qquad\qquad\qquad -\frac{25}{2^6}\,m^4 - \frac{5299}{2^8.3.5}\,m^5,\\
(kk')_{-3} &= \qquad\qquad\qquad\qquad \frac{75}{2^{10}}\,m^5
\end{aligned}$$

correctly to order m^5.

The arithmetical values of these quantities are added.

$$\begin{aligned}
(kk')_{-1} &= +.07279\ \ 42,\\
(kk')_{1} &= -.00282\ \ 74,\\
(kk')_{-2} &= +.00009\ \ 27,\\
(kk')_{2} &= -.00002\ \ 19,\\
(kk')_{-3} &= +.00000\ \ 04,\\
(kk')_{3} &= -.00000\ \ 02.
\end{aligned}$$

In calculating the arithmetical values, we use the equations for $(kk')_j$ and $(kk')_{-j}$ first obtained, as the labor in computing the coefficients is much less than for the equation when $(kk')_{-j}$ has been eliminated. $(kk')_{-j}$ can then be completely eliminated, whereas in the algebraic equation it still has the small coefficient

$$4j^2\{[j] - 8j(-1+j-m)\}\,a_{-2j} + \frac{3m^2}{2}\{[j] - 4(-1+j-m)(1+3j+2m)\}\,a_{2j-1}$$
$$+\frac{3m^2}{2}\{-[j] - 4(-1+j-m)(-1+3j-2m)\}\,a_{-1},$$

the last term being of order m^4.

Moreover the quantities K do not appear in the second of the two simpler equations.

Calculation of the Quantities (kk) *and* $(k'k')$.

On substituting the assumed values of u and s in the differential equations, we obtain from the coefficients of ζ^{2j+2g} the two following equations:

$$\begin{aligned}&\Sigma\{4(j+g)^2+(2i+1-2j)(2i+1+2g)+4m(2i+1-j+g)+\tfrac{9}{2}m^2\}a_{i-j}(kk)_i\\&+\Sigma\{4(j+g)^2+(2i+1+2j)(2i+1-2g)+4m(2i+1+j-g)+\tfrac{9}{2}m^2\}a_{i+j}(k'k')_i\\&\qquad+\tfrac{9}{2}m^2\Sigma a_{-i+j-1}(kk)_i+\tfrac{9}{2}m^2\Sigma a_{-i-j-1}(k'k')_i\\&=\Sigma\{(2i+g)^2+(2j-2i+g)^2+(2i+g)(2j-2i+g)-3m^2\}K_iK_{j-i},\end{aligned}$$

and

$$\begin{aligned}&4(j+g)[\Sigma(2i+1-j+g+m)a_{i-j}(kk)_i+\Sigma(2i+1+j-g+m)a_{i+j}(k'k')_i]\\&\qquad-3m^2\Sigma a_{-i+j-1}(kk)_i+3m^2\Sigma a_{-i-j-1}(k'k')_i=0.\end{aligned}$$

The terms with the largest coefficients are $(kk)_j$, $(k'k')_{-j}$. Eliminating each of these quantities in turn, we get the two equations

$$\begin{aligned}&-4(j+g)\Sigma(j+g-i)[[j]-4i(-1+j+g-m)]a_{-i}(kk)_{j-i}\\&-4(j+g)\Sigma i[[j]-4(j+g-i)(-1+j+g-m)]a_i(k'k')_{-j+i}\\&+\frac{3m^2}{2}[[j]-4(-1+j+g-m)(1+3j+3g+2m)]\Sigma a_{-i+j-1}(kk)_i\\&+\frac{3m^2}{2}[-[j]-4(-1+j+g-m)(-1+3j+3g-2m)]\Sigma a_{-i-j-1}(k'k')_{-j+i}\\&+4(j+g)(-1+j+g-m)\Sigma\{(j-2i)^2+3(j+g)^2-3m^2\}K_iK_{j-i}=0,\end{aligned}$$

and

$$\begin{aligned}&-4(j+g)\Sigma(j+g-i)[[j]-4i(1+j+g+m)]a_i(k'k')_{-j+i}\\&-4(j+g)\Sigma i[[j]-4(j+g-i)(1+j+g+m)]a_{-i}(kk)_{j-i}\\&+\tfrac{3}{2}m^2[[j]-4(1+j+g+m)(-1+3j+3g-2m)]\Sigma a_{-i-j-1}(k'k')_i\\&+\tfrac{3}{2}m^2[-[j]-4(1+j+g+m)(1+3j+3g+2m)]\Sigma a_{-i+j-1}(kk)_i\\&+4(j+g)(1+j+g+m)\Sigma\{(j-2i)^2+3(j+g)^2-3m^2\}K_iK_{j-i}=0,\end{aligned}$$

where $[j]\equiv 8(j+g)^2-2-4m+m^2$.

Solving the equations, we obtain

$$\begin{aligned}(kk)_0&=\frac{3}{2^3}m^2-\frac{51}{2^5}m^3+\frac{543}{2^9}m^4+\frac{2277}{2^{12}}m^5,\\(k'k')_0&=1+0.m-\frac{19}{2^3}m^2+\frac{209}{2^5}m^3-\frac{5109}{2^9}m^4+\frac{89363}{2^{12}.3}m^5,\\(kk)_1&=\frac{25}{2^6}m^4-\frac{5201}{2^9.3.5}m^5,\\(k'k')_{-1}&=\frac{3}{2^4}m^2+\frac{1}{2}m^3+\frac{85}{2^6.3}m^4+\frac{4567}{2^{10}.3^2.5}m^5,\\(kk)_{-1}&=-\frac{3}{2}m+\frac{159}{2^5}m^2-\frac{2451}{2^8}m^3+\frac{41541}{2^{11}}m^4-\frac{25955}{2^{11}}m^5,\end{aligned}$$

$$\begin{aligned}
(k'k')_1 &= \frac{3}{2^2}m - \frac{83}{2^5}m^2 + \frac{67}{2^6.3}m^3 - \frac{228125}{2^{11}.3^2}m^4 - \frac{2450305}{2^{13}.3^3}m^5,\\
(k'k')_{-2} &= \frac{25}{2^8}m^4 + \frac{803}{2^7.3.5}m^5,\\
(kk)_{-2} &= \frac{9}{2^6}m^2 - \frac{27}{2^7}m^3 + \frac{71}{2^{11}}m^4 - \frac{861}{2^{11}}m^5,\\
(k'k')_2 &= \frac{27}{2^6}m^3 - \frac{93}{2^8}m^4 - \frac{13}{2^9.5}m^5,\\
(kk)_{-3} &= \frac{27}{2^{10}}m^4 + \frac{171}{2^{11}}m^5,\\
(k'k')_3 &= \frac{375}{2^{10}}m^5,
\end{aligned}$$

correctly to order m^5.

The arithmetical values are as follows:

$$\begin{aligned}
(kk)_0 &= +.00165\ 68,\\
(kk)_1 &= +.00001\ 41,\\
(kk)_2 &= +.00000\ 01,\\
(kk)_{-1} &= -.09302\ 24,\\
(kk)_{-2} &= +.00080\ 37,\\
(kk)_{-3} &= +.00000\ 14,
\end{aligned}$$

$$\begin{aligned}
(k'k')_0 &= +.98752\ 58,\\
(k'k')_{-1} &= +.00150\ 88,\\
(k'k')_{-2} &= +.00000\ 59,\\
(k'k')_1 &= +.04328\ 71,\\
(k'k')_2 &= +.00020\ 85,\\
(k'k')_3 &= +.00000\ 14.
\end{aligned}$$

A remark similar to that in the case of the quantities (kk') may be made about the arithmetical work.

Comparison with Delaunay's Results.

From the values of u, s an expression for the longitude may be calculated. On comparison with Delaunay's expression (where, however, the symbol m has a different significance), we find that the two expressions are rendered identical if we replace Delaunay's γ^2 by

$$K^2\left(1 - \frac{3}{2^3}m^3 - \frac{141}{2^7}m^4 - \frac{3893}{2^9.3}m^5\right).$$

This relation between the constants of integration of the two theories agrees with the similar relation previously obtained from a comparison of the expressions for the latitude.

Third Powers of the Inclination.

We now return to the z equation,

$$(D^2 - m^2)\, z - \frac{\varkappa z}{(us + z^2)^{\frac{3}{2}}} = 0,$$

but, since the factor $(us + z^2)^{-\frac{3}{2}}$ is inconvenient, we eliminate it by means of one of the other equations of motion,

$$(D^2 + 2m\,D + \tfrac{3}{2}\,m^2)\, u + \tfrac{3}{2}\,m^2 s - \frac{\varkappa u}{(us + z^2)^{\frac{3}{2}}} = 0.$$

Multiplying therefore by u, z and subtracting,

$$D\,(u Dz - z Du) - 2mz\, Du - \tfrac{1}{2}\, m^2 z\,(5u + 3s) = 0,$$

and we assume as the solution

$$\begin{aligned} z\sqrt{-1} = a_0\, K \Sigma K_j\, (\zeta^{2j+g} - \zeta^{-2j-g}) \\ + a_0 K^3\, [\Sigma\, (k^2 k')_j\, (\zeta^{2j+g} - \zeta^{-2j-g}) \\ + \Sigma\, (k^3)_j \quad (\zeta^{2j+3g} - \zeta^{-2j-3g})]. \end{aligned}$$

The motion of the node does not depend solely upon the ratio of the mean motions, and the above equations can only be satisfied (to order K^3) by assuming

$$g = g_0 + g_{\gamma^2} K^2,$$

where g_0 is the value of g already calculated and independent of K, and $g_{\gamma^2} K^2$ the correction depending on the square of the inclination.

Calculation of the Quantities $(k^2 k')_i$.

Substituting in the differential equation, and equating the coefficient of ζ^{2j+1+g} to zero, we obtain

$$\begin{aligned}
&\Sigma\left[(2i+g_0+g_{\gamma^2}K^2)^2-(2j-2i+1)^2-2m(2j-2i+1)-\tfrac{5}{2}m^2\right]K_i a_{j-i}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\tfrac{3}{2}m^2\Sigma K_i a_{i-j-1}\\
&+K^2\left\{\begin{aligned}
&\Sigma\left[(2i+g_0)^2-(2j-2i+1)^2-2m(2j-2i+1)-\tfrac{5}{2}m^2\right]K_i(kk')_{j-i}\\
&\qquad\qquad\qquad\qquad\qquad\qquad -\tfrac{3}{2}m^2\Sigma K_i(kk')_{i-j-1}\\
&+\Sigma\left[-(2i+g_0)^2+(2j+2i+1+2g_0)^2+2m(2j+2i+1+2g_0)+\tfrac{5}{2}m^2\right]K_i(kk)_{j+i}\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\tfrac{3}{2}m^2\Sigma K_i(k'k')_{-i-j-1}\\
&+\Sigma\left[(2i+g_0)^2-(2j-2i+1)^2-2m(2j-2i+1)-\tfrac{5}{2}m^2\right](k^2k')_i a_{j-i}\\
&\qquad\qquad\qquad\qquad\qquad\qquad -\tfrac{3}{2}m^2\Sigma(k^2k')_i a_{i-j-1}
\end{aligned}\right\}=0,
\end{aligned}$$

where Σ includes all integral values of i, positive, negative and zero.

In this equation we may equate coefficients of K^0 and K^2 separately to zero. From the coefficient of K^0 we have

$$\begin{aligned}
&\sum\left[(2i+g_0)^2-(2j-2i+1)^2-2m(2j-2i+1)-\tfrac{5}{2}m^2\right]K_i a_{j-i}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\tfrac{3}{2}m^2\sum_i K_i a_{i-j-1}=0.
\end{aligned}$$

This equation holds for all integral values of j, thus forming a system of equations from which the quantities K may be determined. Now previously the quantities K were determined from the system

$$(2j+g)^2K_j=2\sum_i M_i K_{i+j}$$

for all integral values of j. The two systems must necessarily be equivalent. To show this, we have from the latter system

$$(2i+g)^2K_i a_{j-i}=2a_{j-i}\sum_{j'}M_{j'-i}K_{j'},$$

hence

$$\begin{aligned}
\sum_i(2i+g)^2K_i a_{j-i}&=2\sum_{j'}K_{j'}\sum_i a_{j-i}M_{j'-i}=2\sum_{j'}K_{j'}\sum a_{j-i}M_{i-j'}\\
&=\sum_{j'}K_{j'}\left[\text{coefficient of }\zeta^{2j-2j'+1}\text{ in }\left(\frac{\varkappa}{r^3}+m^2\right)u\right]\\
&=\sum_i K_i\left[\text{coefficient of }\zeta^{2j-2i+1}\text{ in }D^2u+2mDu+m^2u+\frac{3m^2}{2}(u+s)\right]\\
&=\sum_i\left[(2j-2i+1)^2+2m(2j-2i+1)+\tfrac{5}{2}m^2\right]K_i a_{j-i}+\frac{3m^2}{2}\sum K_i a_{i-j-1}.
\end{aligned}$$

Now this is the former system, and the equivalence is established.

Again from the coefficient of K^2,

$$\begin{aligned}
&\Sigma\left[(2i+g_0)^2-(2j-2i+1)^2-2m\,(2j-2i+1)-\tfrac{5}{2}m^2\right]K_i(kk')_{j-i}-\tfrac{3}{2}m^2\Sigma K_i(kk')_{i-j-1}\\
+\,&\Sigma\left[-(2i+g_0)+(2j+2i+1+2g_0)^2+2m\,(2j-2i+1+2g_0)+\tfrac{5}{2}m^2\right]K_i\,(kk)_{j+i}\\
+\,&\tfrac{3}{2}m^2\Sigma K_i\,(k'k')_{-i-j-1}\\
+\,&\Sigma\left[(2i+g_0)^2-(2j-2i+1)^2-2m(2j-2i+1)-\tfrac{5}{2}m^2\right](k^2k')_i a_{j-i}\\
-\,&\tfrac{3}{2}m^2\Sigma\,(k^2k')_i a_{i-j-1}\\
+\,&2g_{\gamma^2}\Sigma\,(2i+g_0)\,K_i a_{j-i}=0\,.
\end{aligned}$$

This is the system of equations from which g_{γ^2} and the quantities (k^2k') must be found.

$(k^2k')_0$ is at our disposal. We make it zero in order to simplify the equations. This is equivalent to defining $2Ka_0$ as the coefficient of the principal periodic term in the value of z correctly as far as the cubes of the inclination. When K has been thus defined, its ratio to Delaunay's γ is determinate to the order of the square of the inclination. Of this ratio the part dependent on m only has already been found.

For a value of j other than zero, the only quantity with a coefficient of order zero is $(k^2k')_j$. The equation (j) therefore determines $(k^2k')_j$. Similarly the equation for which $j=0$ determines g_{γ^2}.

If the quantities (k^2k') be found to the fourth order, the corresponding terms of z are found correctly to the seventh order. This is the degree of accuracy to which we have gone in the first and second powers of the inclination; g_{γ^2} can then be determined to the sixth order. This is one term beyond Delaunay's value. Before however g_{γ^2} is obtained to the sixth order, it is necessary to find $(kk)_0$ to the sixth order. As this has not been done, g_{γ^2} is here given to the fifth order only.

Solving the equations,

$$\begin{aligned}
(k^2k')_{-1} &= -\frac{3}{2^3}m+\frac{85}{2^5}m^2+\frac{281}{2^7.3}m^3+\frac{32509}{2^9.3^2}m^4,\\
(k^2k')_1 &= -\frac{3}{2^3}m^2-\frac{37}{2^6}m^3-\frac{1063}{2^9.3}m^4,\\
(k^2k')_{-2} &= -\frac{27}{2^7}m^3+\frac{183}{2^8}m^4,\\
(k^2k')_2 &= -\frac{25}{2^6}m^4,
\end{aligned}$$

and

$$g_{\gamma^2}=-\frac{3}{2}m^2+\frac{51}{2^4}m^3+\frac{219}{2^7}m^4-\frac{5751}{2^{10}}m^5.$$

If, after substituting the assumed value of z in the differential equation, we had equated to zero the coefficient of ζ^{2j+1-g}, and then proceeded as before, we should have got the equations

$$\begin{aligned}
\Sigma\left[(2i+g_0)^2-(2j+2i+1)^2-2m(2j+2i+1)-\frac{5m^2}{2}\right]K_i(kk')_{i+j} \\
-\frac{3m^2}{2}\Sigma K_i(kk')_{-i-j-1} \\
+\Sigma\left[-(2i+g_0)^2+(2j-2i+1-2g_0)^2+2m(2j-2i+1-2g_0)+\frac{5m^2}{2}\right]K_i(k'k')_{-i+j} \\
+\frac{3m^2}{2}\Sigma K_i(kk)_{i-j-1} \\
+\Sigma\left[(2i+g_0)^2-(2j+2i+1)^2-2m(2j+2i+1)-\frac{5m^2}{2}\right](k^2k')_i a_{i+j} \\
-\frac{3m^2}{2}\Sigma(k^2k')_i a_{-i-j-1} \\
+2g_{\gamma^2}\Sigma(2i+g_0)K_i a_{i+j}=0.
\end{aligned}$$

The quantities (k^2k'), and also g_{γ^2}, may be found from this system of equations, which therefore serve to verify the values already found. Moreover, this system will give g_{γ^2} to the sixth order, without its being necessary to calculate $(kk)_0$ beyond the fifth order.

The value of g_{γ^2} so found is

$$-\frac{3}{2}m^2+\frac{51}{2^4}m^3+\frac{219}{2^7}m^4-\frac{5751}{2^{10}}m^5+\frac{7115}{2^{12}}m^6.$$

The arithmetic values of the quantities are as follows:

$$\begin{aligned}
g_{\gamma^2} &= -.00806\ \ 633, \\
(k^2k')_{-1} &= -.01224\ \ \ \ 0, \\
(k^2k')_1 &= -.00277\ \ \ \ 8, \\
(k^2k')_{-2} &= -.00008\ \ \ \ 0, \\
(k^2k')_2 &= -.00002\ \ \ \ 2, \\
(k^2k')_{-3} &= -.00000\ \ \ \ 1.
\end{aligned}$$

In this case, in the arithmetical work, the same equations are used as in the algebraical.

The expression for the ratio of the motion of the Moon's node to the Moon's mean motion, so far as it consists of a series in m multiplied by the square of the inclination, has been computed by Delaunay to be

$$\gamma^2 \left\{ \frac{3}{2} m'^2 - \frac{27}{16} m'^3 - \frac{843}{128} m'^4 - \frac{7185}{1024} m'^5 \right\},$$

where $m' = \frac{m}{1+m}$, and in this paper to be $\frac{g_{\gamma^2}}{1+m} . K^2$, where

$$g_{\gamma^2} = -\frac{3}{2} m^2 + \frac{51}{2^4} m^3 + \frac{219}{2^7} m^4 - \frac{5751}{2^{10}} m^5 + \frac{7115}{2^{12}} m^6,$$

and where

$$\gamma^2 = K^2 \left(1 - \frac{3}{2^3} m^3 - \frac{141}{2^7} m^4 - \frac{3893}{2^9 . 3} m^5 \dots . \right),$$

or numerically,

$$g_{\gamma^2} = -0.00806\ 633,$$

and the ratio of $\gamma^2 : K^2$ is 0.99974 65, hence

$$-\frac{g_\gamma}{1+m} K^2 = 0.00746\ 483 . \gamma^2.$$

The numerical value of Delaunay's terms is

$$0.00746\ 397\ \gamma^2.$$

The difference is therefore 0.00000 086 γ^2.

Now Delaunay gives 0″7133 as the daily mean motion corresponding to these terms, and therefore the correction does not exceed 0″0001. This is the value of the residue of Delaunay's series that might be inferred from the series itself. The correction is small compared with the discrepancy that exists between theory and observation. No large correction could have been expected from the sequence of Delaunay's terms. From Delaunay's complete expressions for the motion of the perigee and node (Comptes Rendus, LXXIV, p. 29), it appears that the series in m multiplied by γ^2 is of more importance in the expression for motion of the perigee than for the node, whereas the series multiplied by e^2 is of more importance for the motion of the node than for the perigee.

Professor E. W. Brown has given (Amer. Jour., vol. XVII) a method of great analytical interest of calculating new parts of the motion of the perigee and node from the coefficients of the period terms already found; for example, g_{γ^2} can by Professor Brown's methods be deduced, by algebraical processes only, without further reference to the differential equations, from the values of the coefficients a, K, (kk), (kk'), $(k'k')$. The method is an extension of the process whereby Dr. Hill obtained the principal part of the motion of the perigee before calculating any periodic terms other than the variation.

In the equation

$$(D^2 - m^2)\,z - \frac{\varkappa z}{(us + z^2)^{\frac{3}{2}}} = 0$$

we substitute

$$z\sqrt{-1} = \sum_j a_0\,[KK_j + K^3(K^2K')_j]\,[\zeta^{2j+g} - \zeta^{-2j-g}],$$

where $g = g_0 + g_{\gamma^2}K^2$, and where, in the value of z, terms containing $3g$ in the index are omitted, as they do not affect what follows. Then equating to zero the coefficient of $K^3.\zeta^{2j+g}$, and employing the symbols M defined by

$$\tfrac{1}{2}\left(\frac{\varkappa}{r^3} + m^2\right) = \sum_{-\infty}^{\infty} M_i\,\zeta^{2i},$$

we have

$$2\,(2j + g_0)\,g_{\gamma^2}K_j + (2j + g_0)^2\,(k^2k')_j - 2\Sigma\,M_{j-i}\,(k^2k')_i$$
$$= \text{coefficient of } K^3.\zeta^{2j+g} \text{ in } \frac{kz_\gamma\sqrt{-1}}{a_0\{(u_0 + u_{\gamma^2})(s_0 + s_{\gamma^2}) + z_\gamma^2\}^{\frac{3}{2}}}.$$

Multiplying by K_j and summing for all values of j, the coefficient of $(k^2k')_i$ arising from the second and third terms on the left-hand side is

$$(2i + g_0)^2\,K_i - 2\Sigma\,M_{j-i}\,K_j,$$

which is zero.

Since $-K_j$ is the coefficient of $K\zeta^{-2j-g}$ in the value of $z_\gamma\sqrt{-1}/a_0$, on the right-hand side we get *half* the coefficient of K^4 in the constant part of the value of $\frac{kz_\gamma^2}{a_0^2\{u_0s_0 + u_{\gamma^2}s_0 + s_{\gamma^2}u_0 + z_\gamma^2\}^{\frac{3}{2}}}$, the presence of the factor one-half being due to the fact that the same terms arise on considering the coefficients of $K^3\zeta^{-2j-g}$ and $K\zeta^{2j+g}$. Hence g_{γ^2} is given by the equation

$$4g_{\gamma^2}\Sigma\,(2j + g)^2\,K_{j^2} = \text{const. coeff. of } K^4 \text{ in } \frac{\varkappa z_\gamma^2}{a_0^2\{u_0s_0 + u_{\gamma^2}s_0 + s_{\gamma^2}u_0 + z_\gamma^2\}^{\frac{3}{2}}}.$$

Were g_{γ^2} determined in this manner, the approximations required to determine the coefficients (k^2k') would be largely reduced. Moreover the equation that previously determined g_{γ^2} would serve as a useful equation of verification.

But in the first place it may be noticed that this method will only give g_{γ^2} algebraically to the same order that the quantities (kk), (kk'), $(k'k')$ have been calculated, whereas the solution by continued approximation gives its value to one higher order. Possibly a similar objection would hold in a numerical application of the method. Again the arbitrary constants of integration arose from differential equations from which $\varkappa$, the mass of the Earth and Moon, had been eliminated. $\varkappa$ is therefore a function of these constants. On p. 145, Amer. Jour., vol. I, Dr. Hill gives

$$\mu = \varkappa (n - n')^2 = n^2 a_0^3 \left[1 - \tfrac{1}{6} m^2 + \ldots . \right]^{-3},$$

but in the higher approximation proceeded to in this paper some terms proportional to K^2 would have to be added to this value. These terms have not been calculated, and it would be unnecessary to calculate them unless it were to apply Prof. Brown's method of computing g_{γ^2}. It would be possible, of course, to replace the arbitrary condition $(kk')_0 = 0$ by the condition that μ shall contain no terms proportional to K^2, but this is merely throwing a heavy piece of work upon an earlier portion of the paper. Moreover, when this has been done, or when the correction to μ has been calculated (whichever alternative is adopted), there still remains the labor of multiplying several series of powers of ζ. The process of solving by continued approximation was indeed long, but it is extremely doubtful whether the practical application of Prof. Brown's method (to the motion of the node at any rate) would not be longer. It must be remembered, too, that the coefficients (k^2k') would still remain to be calculated.

Calculation of the Quantities (k^3).

Substituting in the differential equation, and equating the coefficients of $\zeta^{2j+1\pm 3g}$ to zero, we obtain the equations

$$\begin{aligned}
&\sum_i \left[(3g_0 + 2i)^2 - (2j - 2i + 1)^2 - 2m(2j - 2i + 1) - \tfrac{5}{2} m^2\right] a_{j-i} (k^3)_i \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \tfrac{3}{2} m^2 \sum_i a_{i-j-1} (k^3)_i \\
&+ \sum_i \left[(g_0 + 2i)^2 - (2j - 2i + 1 + 2g_0)^2 - 2m(2j - 2i + 1 + 2g_0) - \tfrac{5}{2} m^2\right] K_i (kk)_{j-i} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \tfrac{3}{2} m^2 \sum_i K_i (k'k')_{i-j-1} = 0,
\end{aligned}$$

and

$$\sum_i \left[(3g_0+2i)^2-(2i+2j+1)^2-2m(2i+2j+1)-\tfrac{5}{2}m^2\right] a_{i+j}\,(k^3)_i - \tfrac{3}{2}m^2\sum_i a_{-i-j-1}\,(k^3)_i$$

$$+\sum_i \left[(g_0+2i)^2-(2j+2i+1-2g_0)^2-2m(2j+2i+1-2g_0)-\tfrac{5}{2}m^2\right] K_i\,(k'k')_{i+j} - \tfrac{3}{2}m^2\sum_i K_i\,(kk)_{-i-j-1} = 0.$$

In the first of these equations $(k^3)_j$, and in the second $(k^3)_{-j}$, are the only quantities that occur with coefficients of order zero. Hence the equations, as they stand, serve to determine $(k^3)_j$ and $(k^3)_{-j}$ respectively.

In either equation j can be given all integral values, positive and negative. Hence either system will determine all the quantities (k^3); the values of these quantities can then be verified by the other system of equations.

In this way the following results have been obtained:

$$\begin{aligned}
(k^3)_{-1} &= -\frac{9}{2^3}m + \frac{117}{2^5}m^2 - \frac{1575}{2^8}m^3 + \frac{26943}{2^{11}}m^4,\\
(k^3)_0 &= \frac{3}{2^3}m^2 - \frac{195}{2^7}m^3 + \frac{405}{2^9}m^4,\\
(k^3)_{-2} &= -\frac{9}{2^5}m^2 + \frac{117}{2^8}m^3 - \frac{1607}{2^{12}}m^4,\\
(k^3)_1 &= \frac{25}{2^6}m^4,\\
(k^3)_{-3} &= -\frac{81}{2^9}m^4.
\end{aligned}$$

The arithmetical results are given below.

$$\begin{aligned}
(k^3)_{-1} &= -.06974\ 0,\\
(k^3)_0 &= +.00168\ 5,\\
(k^3)_{-2} &= -.00159\ 9,\\
(k^3)_1 &= +.00001\ 4,\\
(k^3)_{-3} &= -.00000\ 6.
\end{aligned}$$

As before, the system of equations was first reduced to a linear system with numerical coefficients. The accuracy required in the coefficient was estimated from the approximate values of the quantities p as given by their algebraic values.

This completes the calculation of the terms involving the first three orders of the inclination.